文章编号:1671-5942(2012)06-0081-05

测边网坐标的总体最小二乘平差方法*

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摘 要 测边网坐标平差中由于已知点含有误差,因而其误差方程的系数矩阵含有误差且与边长观测值相关并精度不等。推导了该条件下的总体最小二乘平差公式,并利用矩阵拉直运算进行随机模型的求解,给出了总体最小二乘平差的精度评定公式。最后通过算例验证了该方法的有效性和可行性。

关键词 总体最小二乘平差;矩阵拉直运算;方差-协方差矩阵;权矩阵; 测边网

中图分类号:P207 **文献标识码**:A

TRILATERATION NET'S COORDINATE ADJUSTMENT BASED ON TOTAL LEAST SQUARES

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Abstract The calculation adjustment formulas based on total least squares (TLS) is deduced. In total least squares adjustment(TLSA), the stochastic model is solved by matrix vector operator. Different from the published research, the observations and elements in coefficient matrix are heteroscedastic and correlated. In trilateration net, the known points are obtained from higher grade surveying and adjustment, so the coordinates of these known points have errors. Furthermore, the coefficient matrix of trilateration net's error equation has errors and is correlated to the observations. Seeing that, the trilateration net total least squares coordinates adjustment is presented. At last, through an example, the method is discussed, and some conclusions are drawn.

Key words: total least squares adjustment(TLSA); matrix vector operator; variance-covariance matrix; weight matrix; trilateration net

1 引言

如何处理系数矩阵和观测向量同时存在的误差,是数据处理领域研究的新课题。总体最小二乘方法(TLS)是最近三十年发展起来的一种能同时顾及观测值误差和模型系数矩阵误差的数学方法,在数学界得到了广泛的应用[1-6],但是在测量数据处理中应用还不是很多。目前,国内外已有的研究都是基于观测值与系数矩阵元素之间是独立的[7-16],或仅考虑等权的情况[13-16],观测值与系数矩阵元素相关且不等精度情况下的总体最小二乘平差方法研究鲜有报道。同时,已有研究中总体最小二乘方法都是应用于直线拟合[7,13,17]、平面拟合[9]、曲线拟合[10]、空间后方交会[16]、坐标转换[8,11,12,14]、应变参

* **收稿日期**:2012-04-01

基金项目:国家自然科学基金(41204003);东华理工大学博士科研启动基金(DHBK201113);江西省数字国土重点实验室开放研究基金(DLLJ201207)

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数反演[18]等系数矩阵比较简单的情况；已有总体最小二乘问题的计算方法主要分为两大类：基于奇异值分解（SVD）的方法[1-6,11,12,16]和Schaffrin等[7-10,13-15]发展的基于拉格朗日求极值的迭代法，这两种方法都存在着计算复杂、很难被测绘人员接受等特点。针对这些问题，本文将以一种新的思路和方法将总体最小二乘问题的求解转换为传统的、为大家熟知的最小二乘平差，详细推导了模型中观测值与系数矩阵元素是相关的、且精度不等情况下的总体最小二乘平差计算公式，并应用于测边网坐标平差中，通过算例验证了方法的有效性和可行性。

2 总体最小二乘平差方法

2.1 模型参数的总体最小二乘估计

平差问题建立的函数模型是

$$(\boldsymbol{B}+\boldsymbol{E}_B)\boldsymbol{X}=\boldsymbol{L}+\boldsymbol{e} \tag{1}$$

式中，$\boldsymbol{L}\in\boldsymbol{R}^{n\times 1}$为观测向量；$\boldsymbol{e}$为观测向量的随机误差；$\boldsymbol{B}\in\boldsymbol{R}^{n\times m}$为列满秩设计矩阵；$\boldsymbol{E}_B$为设计矩阵的误差；$\boldsymbol{X}$为待估参数向量。

$$\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{e}_B\end{bmatrix}=\begin{bmatrix}e\\ \mathrm{vec}(\boldsymbol{E}_B)\end{bmatrix}\sim\left(\begin{bmatrix}0\\0\end{bmatrix},\sigma_0^2\begin{bmatrix}\boldsymbol{Q}_{ee} & \boldsymbol{Q}_{ee_B}\\ \boldsymbol{Q}_{e_Be} & \boldsymbol{Q}_{e_Be_B}\end{bmatrix}\right) \tag{2}$$

式中，$\boldsymbol{e}_B$是将矩阵$\boldsymbol{E}_B$按列拉直得到的列向量；Q_{ee}为观测值的协因数阵；$\boldsymbol{Q}_{e_Be_B}$为系数矩阵元素的协因数阵；$\boldsymbol{Q}_{ee_B}$为观测值与系数矩阵元素的互协因数阵，$\boldsymbol{Q}_{ee_B}=\boldsymbol{Q}_{e_Be}^{\mathrm{T}}$。

当观测值与系数矩阵元素独立（不相关）同分布（等精度）时，有

$$\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{e}_B\end{bmatrix}=\begin{bmatrix}\boldsymbol{e}\\ \mathrm{vec}(\boldsymbol{E}_B)\end{bmatrix}\sim\left(\begin{bmatrix}0\\0\end{bmatrix},\sigma_0^2\begin{bmatrix}\boldsymbol{I}_n & 0\\ 0 & \boldsymbol{I}_m\otimes\boldsymbol{I}_n\end{bmatrix}\right) \tag{3}$$

本文主要研究的是观测值和系数矩阵元素相关且精度不等情况下的总体最小二乘问题，即平差问题具有式(2)所示的随机模型。

由式(1)得

$$\bar{\boldsymbol{e}}=\boldsymbol{e}-\boldsymbol{E}_B\boldsymbol{X}=\boldsymbol{B}\boldsymbol{X}-\boldsymbol{L} \tag{4}$$

根据观测值与系数矩阵元素相关、且精度不等情况下的总体最小二乘准则

$$\bar{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{P}_{\bar{e}}\bar{\boldsymbol{e}}=\min \tag{5}$$

求解目标函数的自由极值得

$$\frac{\partial\hat{\bar{\boldsymbol{e}}}^{\mathrm{T}}\boldsymbol{P}_{\bar{e}}\hat{\bar{\boldsymbol{e}}}}{\partial\hat{\boldsymbol{X}}}=0 \tag{6}$$

由于

$$\bar{\boldsymbol{e}}=\boldsymbol{e}-\boldsymbol{E}_B\boldsymbol{X}=\begin{bmatrix}e_1\\e_2\\ \vdots\\ e_n\end{bmatrix}-\begin{bmatrix}e_{B_{11}} & e_{B_{12}} & \cdots & e_{B_{1m}}\\ e_{B_{21}} & e_{B_{22}} & \cdots & e_{B_{2m}}\\ \vdots & \vdots & \vdots & \vdots\\ e_{B_{n1}} & e_{B_{n2}} & \cdots & e_{B_{nm}}\end{bmatrix}\begin{bmatrix}X_1\\X_2\\ \vdots\\ X_m\end{bmatrix}=$$

$$\begin{bmatrix}e_1-X_1e_{B_{11}}-X_2e_{B_{12}}-\cdots-X_me_{B_{1m}}\\ e_2-X_1e_{B_{21}}-X_2e_{B_{22}}-\cdots-X_me_{B_{2m}}\\ \vdots\\ e_n-X_1e_{B_{n1}}-X_2e_{B_{n2}}-\cdots-X_me_{B_{nm}}\end{bmatrix}=\boldsymbol{K}\mathrm{vec}(\boldsymbol{G}) \tag{7}$$

式中，$\boldsymbol{K}=$

$$\begin{bmatrix}1 & -X_1 & -X_2 & \cdots & -X_m & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 1 & -X_1 & -X_2 & \cdots & -X_m\end{bmatrix},$$

$\boldsymbol{K}\in\boldsymbol{R}^{n\times(n\times m+n)}$；$\boldsymbol{G}=\begin{bmatrix}e_1 & e_{B_{11}} & e_{B_{12}} & \cdots & e_{B_{1m}}\\ e_2 & e_{B_{21}} & e_{B_{22}} & \cdots & e_{B_{2m}}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ e_n & e_{B_{n1}} & e_{B_{n2}} & \cdots & e_{B_{nm}}\end{bmatrix}$，

$\boldsymbol{G}\in\boldsymbol{R}^{n\times(m+1)}$；$\mathrm{vec}(\boldsymbol{G})\in\boldsymbol{R}^{(n\times m+n)\times 1}$。$\mathrm{vec}(\boldsymbol{G})$是指将矩阵$\boldsymbol{G}$按行拉直所得到的列向量，排列的顺序为从上到下。

在实际应用中，大部分平差模型的观测值与系数矩阵元素之间是相关且精度不等的，即矩阵$\boldsymbol{G}$中的元素是相关的且精度不等的，而已有的处理等权、不相关情况下的总体最小二乘方法不再适用；在这种情况下，如果已知列向量$\mathrm{vec}(\boldsymbol{G})$中的所有元素$e_1,e_{B_{11}},e_{B_{12}},\cdots,e_{B_{nm}}$之间的互协因数及协因数组成的矩阵（$\boldsymbol{Q}_{vec(\boldsymbol{G})}\in\boldsymbol{R}^{(n\times m+n)\times(n\times m+n)}$）。由式(7)根据协因数传播律[19]得

$$\boldsymbol{Q}_{\bar{e}}=\boldsymbol{K}\boldsymbol{Q}_{vec(\boldsymbol{G})}\boldsymbol{K}^{\mathrm{T}} \tag{8}$$

所以有

$$\frac{\partial\hat{\bar{\boldsymbol{e}}}^{\mathrm{T}}\boldsymbol{P}_{\bar{e}}\hat{\bar{\boldsymbol{e}}}}{\partial\hat{\boldsymbol{X}}}=2\hat{\bar{\boldsymbol{e}}}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{B}+2\hat{\bar{\boldsymbol{e}}}^{\mathrm{T}}\hat{\bar{\boldsymbol{e}}}\frac{\partial\boldsymbol{Q}_{\bar{e}}^{-1}}{\partial\hat{\boldsymbol{X}}}=$$

$$2\hat{\bar{\boldsymbol{e}}}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{B}+2\hat{\bar{\boldsymbol{e}}}^{\mathrm{T}}\hat{\bar{\boldsymbol{e}}}\frac{\partial(\boldsymbol{K}\boldsymbol{Q}_{vec(\boldsymbol{G})}\boldsymbol{K}^{\mathrm{T}})^{-1}}{\partial\hat{\boldsymbol{X}}}=0 \tag{9}$$

由于$\hat{\bar{\boldsymbol{e}}}^{\mathrm{T}}\hat{\bar{\boldsymbol{e}}}$是一个小量，在计算过程中舍掉。利用最小二乘计算的结果代入(7)式，得到$\boldsymbol{K}$矩阵，从而求得$\boldsymbol{Q}_{\bar{e}}$。将式(4)和式(8)代入式(9)，得参数的TLS估值为

$$\hat{\boldsymbol{X}}=(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{L} \tag{10}$$

对于TLS中关键的因素$\boldsymbol{Q}_{\bar{e}}$可以采用$\boldsymbol{Q}(\boldsymbol{e},\boldsymbol{e}_B)$矩阵分块进行求解：

$$\bar{\boldsymbol{e}}=[\boldsymbol{I}_n \quad -\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n]\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{e}_B\end{bmatrix}=\boldsymbol{e}-\boldsymbol{E}_B\boldsymbol{X} \tag{11}$$

式中，$\boldsymbol{e}_B=\mathrm{vec}(\boldsymbol{E}_B)$，$\boldsymbol{e}_B$是将矩阵$\boldsymbol{E}_B$按列拉直得到的列向量，$\boldsymbol{e}_B\in\boldsymbol{R}^{(n\times m)\times 1}$；$\otimes$表示Kronecker积，$[\boldsymbol{I}_n \quad -\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n]\in R^{n\times(n\times m+n)}$，$[\boldsymbol{I}_n \quad -\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n]=$
$[\boldsymbol{I}_n \quad -\boldsymbol{X}_1\boldsymbol{I}_n \quad \cdots \quad -\boldsymbol{X}_m\boldsymbol{I}_n]=$

$$\begin{bmatrix}1 & & & -X_1 & & \cdots & -X_m & \\ & \ddots & & & \ddots & \cdots & & \ddots\\ & & 1 & & -X_1 & \cdots & & -X_m\end{bmatrix} \tag{12}$$

若已知观测值误差 $\boldsymbol{e}$ 与系数阵误差阵 $\boldsymbol{E}_B$ 拉直后的列向量 $\boldsymbol{e}_B$ 的协因数 $\boldsymbol{Q}_{ee}$、$\boldsymbol{Q}_{e_Be_B}$ 及互协因数阵 $\boldsymbol{Q}_{ee_B}$,则由式(11)根据协因数传播律得:

$$Q_{\bar{e}}=\left[\boldsymbol{I}_n \quad -\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n\right]\begin{bmatrix}\boldsymbol{Q}_{ee} & \boldsymbol{Q}_{ee_B}\\ \boldsymbol{Q}_{e_Be} & \boldsymbol{Q}_{e_Be_B}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}_n\\ -\boldsymbol{X}\otimes\boldsymbol{I}_n\end{bmatrix}$$

$$=\left[\boldsymbol{Q}_{ee}-(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_{e_Be} \quad \boldsymbol{Q}_{ee_B}-(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_{e_Be_B}\right]\begin{bmatrix}\boldsymbol{I}_n\\ -\boldsymbol{X}\otimes\boldsymbol{I}_n\end{bmatrix}$$

$$=\boldsymbol{Q}_{ee}-(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_{e_Be}-\boldsymbol{Q}_{ee_B}(\boldsymbol{X}\otimes\boldsymbol{I}_n)+(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_{e_Be_B}(\boldsymbol{X}\otimes\boldsymbol{I}_n) \tag{13}$$

因为 $\boldsymbol{Q}_{ee_B}=\boldsymbol{Q}_{e_Be}^{\mathrm{T}}$,$(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)^{\mathrm{T}}=(\boldsymbol{X}\otimes\boldsymbol{I}_n)$,所以

$$\left[(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_{e_Be}\right]^{\mathrm{T}}=\boldsymbol{Q}_{ee_B}(\boldsymbol{X}\otimes\boldsymbol{I}_n) \tag{14}$$

则式(13)可以写成

$$\boldsymbol{Q}_{\bar{e}}=\boldsymbol{Q}_{ee}-(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_{e_Be}-\left[(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_{e_Be}\right]^{\mathrm{T}}+(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)\boldsymbol{Q}_{e_Be_B}(\boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{I}_n)^{\mathrm{T}} \tag{15}$$

在 $\boldsymbol{Q}_{\bar{e}}$(或 $\boldsymbol{Q}_{\bar{e}}^{-1}$)的计算式(8)和(15)中,$\boldsymbol{X}$ 及 $\boldsymbol{X}^{\mathrm{T}}$ 都是采用的最小二乘估值 $\hat{\boldsymbol{X}}_{\mathrm{LS}}$ 及 $\hat{\boldsymbol{X}}_{\mathrm{LS}}^{\mathrm{T}}$。

2.2　总体最小二乘平差的精度评定

单位权方差的估计为

$$\hat{\sigma}_0^2=\frac{\hat{\bar{\boldsymbol{e}}}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\hat{\bar{\boldsymbol{e}}}}{n-m} \tag{16}$$

式中,$\hat{\bar{\boldsymbol{e}}}=\boldsymbol{B}\hat{\boldsymbol{X}}-\boldsymbol{L}=\left[\boldsymbol{B}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}-\boldsymbol{I}_n\right]\boldsymbol{L}$

协因数阵为

$$\boldsymbol{Q}_{\hat{X}\hat{X}}=(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{Q}_{e}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{B}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Q}_{\bar{e}}^{-1}\boldsymbol{B})^{-1} \tag{17}$$

3　测边网坐标的总体最小二乘平差方法

如图 1 所示,点 A、B、C 为已知点,L_1、L_2、L_3 为观测边,D 为待定点。则边 AD 的误差方程为[19]

$$v_{AD}=\frac{\Delta X_{AD}}{S_{AD}}\hat{x}_D+\frac{\Delta Y_{AD}}{S_{AD}}\hat{y}_D-l_{AD} \tag{18}$$

式中,$\Delta X_{AD}=X_D-X_A$,$\Delta Y_{AD}=Y_D-Y_A$,$S_{AD}=\sqrt{(X_D-X_A)^2+(Y_D-Y_A)^2}$,$l_{AD}=L_1-S_{AD}$

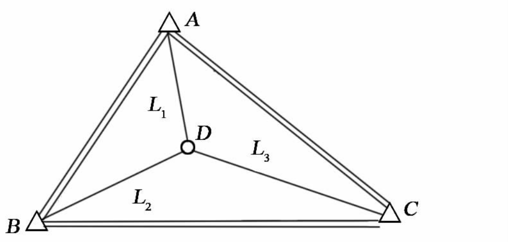

图 1　测边网示意图[19]

Fig. 1　Sketch of trilateration network[19]

列出所有的误差方程后,得到误差方程 $\boldsymbol{V}=\boldsymbol{B}\hat{\boldsymbol{x}}-\boldsymbol{l}$,$\boldsymbol{B}\in\boldsymbol{R}^{3\times2}$ 为系数矩阵,$\hat{\boldsymbol{x}}$ 为 D 点坐标改正数向量。假设矩阵 $\boldsymbol{B}$ 的元素为 B_{ij}($i=1,2,3$;$j=1,2$),即[19]

$$\boldsymbol{B}=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\\ B_{31} & B_{32}\end{bmatrix}=\begin{bmatrix}\dfrac{\Delta X_{AD}}{S_{AD}} & \dfrac{\Delta Y_{AD}}{S_{AD}}\\ \dfrac{\Delta X_{BD}}{S_{BD}} & \dfrac{\Delta Y_{BD}}{S_{BD}}\\ \dfrac{\Delta X_{CD}}{S_{CD}} & \dfrac{\Delta Y_{CD}}{S_{CD}}\end{bmatrix} \tag{19}$$

式中 B_{11}、B_{12} 分别为:

$$B_{11}=\frac{\Delta X_{AD}}{S_{AD}}=\frac{1}{2L_1}\frac{1}{\overline{AB}^2}\left((L_1^2+\overline{AB}^2-L_2^2)(X_B-X_A)+(4L_1^2\overline{AB}^2-(L_1^2+\overline{AB}^2-L_2^2)^2)^{\frac{1}{2}}(Y_B-Y_A)\right) \tag{20}$$

$$B_{12}=\frac{\Delta Y_{AD}}{S_{AD}}=\frac{1}{2L_1}\frac{1}{\overline{AB}^2}\left((L_1^2+\overline{AB}^2-L_2^2)(Y_B-Y_A)+(4L_1^2\overline{AB}^2-(L_1^2+\overline{AB}^2-L_2^2)^2)^{\frac{1}{2}}(X_B-X_A)\right) \tag{21}$$

系数矩阵中元素 B_{11}、B_{12} 是 L_1、L_2、X_A、Y_A、X_B、Y_B 的函数,因而分别求偏导,其结果见附录;同理可以求得 B_{21}、B_{22} 关于 L_2、L_3、X_B、Y_B、X_C、Y_C 的偏导数 $\frac{\partial B_{21}}{\partial L_2}$、$\frac{\partial B_{21}}{\partial L_3}$、$\frac{\partial B_{21}}{\partial X_B}$、$\frac{\partial B_{21}}{\partial Y_B}$、$\frac{\partial B_{21}}{\partial X_C}$、$\frac{\partial B_{21}}{\partial Y_C}$ 及 $\frac{\partial B_{22}}{\partial L_2}$、$\frac{\partial B_{22}}{\partial L_3}$、$\frac{\partial B_{22}}{\partial X_B}$、$\frac{\partial B_{22}}{\partial Y_B}$、$\frac{\partial B_{22}}{\partial X_C}$、$\frac{\partial B_{22}}{\partial Y_C}$;$B_{31}$、$B_{32}$ 关于 L_1、L_3、X_A、Y_A、X_C、Y_C 的偏导数 $\frac{\partial B_{31}}{\partial L_1}$、$\frac{\partial B_{31}}{\partial L_3}$、$\frac{\partial B_{31}}{\partial X_A}$、$\frac{\partial B_{31}}{\partial Y_A}$、$\frac{\partial B_{31}}{\partial X_C}$、$\frac{\partial B_{31}}{\partial Y_C}$ 及 $\frac{\partial B_{32}}{\partial L_1}$、$\frac{\partial B_{32}}{\partial L_3}$、$\frac{\partial B_{32}}{\partial X_A}$、$\frac{\partial B_{32}}{\partial Y_A}$、$\frac{\partial B_{32}}{\partial X_C}$、$\frac{\partial B_{32}}{\partial Y_C}$。

所以有:$\boldsymbol{\Xi}=$

$$\left.\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{\partial B_{11}}{\partial L_1} & \frac{\partial B_{11}}{\partial L_2} & 0 & \frac{\partial B_{11}}{\partial X_A} & \frac{\partial B_{11}}{\partial Y_A} & \frac{\partial B_{11}}{\partial X_B} & \frac{\partial B_{11}}{\partial Y_B} & 0 & 0\\
\frac{\partial B_{12}}{\partial L_1} & \frac{\partial B_{12}}{\partial L_2} & 0 & \frac{\partial B_{12}}{\partial X_A} & \frac{\partial B_{12}}{\partial Y_A} & \frac{\partial B_{12}}{\partial X_B} & \frac{\partial B_{12}}{\partial Y_B} & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{\partial B_{21}}{\partial L_2} & \frac{\partial B_{21}}{\partial L_3} & 0 & 0 & \frac{\partial B_{21}}{\partial X_B} & \frac{\partial B_{21}}{\partial Y_B} & \frac{\partial B_{21}}{\partial X_C} & \frac{\partial B_{21}}{\partial Y_C}\\
0 & \frac{\partial B_{22}}{\partial L_2} & \frac{\partial B_{22}}{\partial L_3} & 0 & 0 & \frac{\partial B_{22}}{\partial X_B} & \frac{\partial B_{22}}{\partial Y_B} & \frac{\partial B_{22}}{\partial X_C} & \frac{\partial B_{22}}{\partial Y_C}\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{\partial B_{31}}{\partial L_1} & 0 & \frac{\partial B_{31}}{\partial L_3} & \frac{\partial B_{31}}{\partial X_A} & \frac{\partial B_{31}}{\partial Y_A} & 0 & 0 & \frac{\partial B_{31}}{\partial X_C} & \frac{\partial B_{31}}{\partial Y_C}\\
\frac{\partial B_{32}}{\partial L_1} & 0 & \frac{\partial B_{32}}{\partial L_3} & \frac{\partial B_{32}}{\partial X_A} & \frac{\partial B_{32}}{\partial Y_A} & 0 & 0 & \frac{\partial B_{32}}{\partial X_C} & \frac{\partial B_{32}}{\partial Y_C}
\end{bmatrix}\right|_{L,X,Y} \tag{22}$$

式中,$L(L_1,L_2,L_3)$ 的取值为观测值,X、$Y(X_A,Y_A,$

X_B, Y_B, X_C, Y_C)的值为高一等级测量后经平差得到的已知点的坐标值。

$$\boldsymbol{\delta\varphi} = \boldsymbol{\Xi h} \tag{23}$$

式中,$\boldsymbol{\delta\varphi} = [dL_1\ dB_{11}\ dB_{12}\ dL_2\ dB_{21}\ dB_{22}\ dL_3\ dB_{31}\ dB_{32}]^{\mathrm{T}}$,$\boldsymbol{h} = [dL_1\ dL_2\ dL_3\ dX_A\ dY_A\ dX_B\ dY_B\ dX_C\ dY_C]^{\mathrm{T}}$

当已知观测值 $L(L_1, L_2, L_3)$ 的协因数及 X、Y $(X_A, Y_A, X_B, Y_B, X_C, Y_C)$ 的协因数 $\boldsymbol{Q}$ 有:

$$\boldsymbol{Q} = \begin{bmatrix} Q_{L_1L_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{L_2L_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{L_3L_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{X_AX_A} & Q_{X_AY_A} & Q_{X_AX_B} & Q_{X_AY_B} & Q_{X_AX_C} & Q_{X_AY_C} \\ 0 & 0 & 0 & Q_{Y_AX_A} & Q_{Y_AY_A} & Q_{Y_AX_B} & Q_{Y_AY_B} & Q_{Y_AX_C} & Q_{Y_AY_C} \\ 0 & 0 & 0 & Q_{X_BX_A} & Q_{X_BY_A} & Q_{X_BX_B} & Q_{X_BY_B} & Q_{X_BX_C} & Q_{X_BY_C} \\ 0 & 0 & 0 & Q_{Y_BX_A} & Q_{Y_BY_A} & Q_{Y_BX_B} & Q_{Y_BY_B} & Q_{Y_BX_C} & Q_{Y_BY_C} \\ 0 & 0 & 0 & Q_{X_CX_A} & Q_{X_CY_A} & Q_{X_CX_B} & Q_{X_CY_B} & Q_{X_CX_C} & Q_{X_CY_C} \\ 0 & 0 & 0 & Q_{Y_CX_A} & Q_{Y_CY_A} & Q_{Y_CX_B} & Q_{Y_CY_B} & Q_{Y_CX_C} & Q_{Y_CY_C} \end{bmatrix} \tag{24}$$

所以有

$$\boldsymbol{Q}_{L,B_{11}\cdots B_{33}} = \boldsymbol{\Xi Q \Xi}^{\mathrm{T}} \tag{25}$$

$\boldsymbol{Q}_{L,B_{11}\cdots B_{33}}$ 即对应于式(8)的 $\boldsymbol{Q}_{\mathrm{vec}(\boldsymbol{G})}$,根据式(10)即可得到未知点 D 的 TLS 坐标估值;根据式(16)和(17)即可进行精度评定。

4 算例及分析

将文献[19]中的例 5-3 进行改化。同精度测得如图 1 所示的三个边长,其结果为 $L_1 = 387.363$ m,$L_2 = 306.065$ m,$L_3 = 354.862$ m,测边中误差为 5 cm,单位权中误差为 1 cm。假设 A、B、C 三点是由更高等级控制网平差得到的,在此作为具有误差的已知点使用,三点的坐标平差值见文献[19]中的表 5-6,其平差后的协因数阵如表 1 所示,单位权中误差为 1 cm。

表 1 已知点坐标协因数数据表

Tab. 1 Coordinated factor data of known points coordinates

	X_A	Y_A	X_B	Y_B	X_C	Y_C
X_A	5.401	-0.413	0.780	-0.832	2.015	-0.066
Y_A	-0.413	6.340	-0.948	3.881	-0.179	2.491
X_B	0.780	-0.948	5.935	-1.813	1.902	-2.094
Y_B	-0.832	3.881	-1.813	8.361	-1.511	2.793
X_C	2.015	-0.179	1.902	-1.511	4.837	-0.532
Y_C	-0.066	2.491	-2.094	2.793	-0.532	8.031

已知点 A、B、C 的起算数据(坐标)和边长观测值含有误差,因而计算得到的系数矩阵 $\boldsymbol{B}$ 含有误差,$\boldsymbol{B} = \begin{bmatrix} -0.9447 & 0.3279 \\ 0.7629 & 0.6465 \\ 0.3262 & -0.9453 \end{bmatrix}$,$\boldsymbol{W} = [0\quad 0\quad 0.231]^{\mathrm{T}}$。令 $\boldsymbol{G} = \begin{bmatrix} e_{l_1} & e_{B_{11}} & e_{B_{12}} \\ e_{l_2} & e_{B_{21}} & e_{B_{22}} \\ e_{l_3} & e_{B_{31}} & e_{B_{32}} \end{bmatrix}$,假设 $\boldsymbol{Q}$ 为列向量中 $\mathrm{vec}(\boldsymbol{G})$ 的所有元素 $e_{l_1}, e_{B_{11}}, e_{B_{12}}, \cdots, e_{B_{32}}$ 之间的互协因数及协因数组成的矩阵,系数矩阵 $\boldsymbol{B}$ 中的元素是由 $\frac{\Delta X_{AD}}{S_{AD}}$、$\frac{\Delta Y_{AD}}{S_{AD}}$、$\frac{\Delta X_{BD}}{S_{BD}}$、$\frac{\Delta Y_{BD}}{S_{BD}}$、$\frac{\Delta X_{CD}}{S_{CD}}$ 及 $\frac{\Delta Y_{CD}}{S_{CD}}$ 组成,这些元素是 L_1、L_2、L_3、X_A、Y_A、X_B、Y_B、X_C、Y_C 的函数,因此利用协因数传播律可计算得到 $\boldsymbol{Q}$ 矩阵。以最小二乘计算结果作为总体最小二乘方法中的初值,则可以得到 $\boldsymbol{K}$ 矩阵,并进一步得到观测值与系数矩阵元素组合观测值的协因数阵 $\boldsymbol{Q}_{\bar{e}} = \begin{bmatrix} 2343.1100 & 456.5174 & -830.1696 \\ 456.5174 & 426.7880 & -225.5155 \\ -830.1696 & -225.5155 & 2636.1549 \end{bmatrix}$。

计算结果见表 2。通过该算例说明了本文推导的总体最小二乘平差方法的应用,由于系数阵中的元素都是已知点坐标和边长观测值的函数,算例中根据已知点 A、B、C 坐标的误差以及边长观测值的误差利用误差传播定律推导了 $\mathrm{vec}(\boldsymbol{G})$ 中的所有元素 e_{L_1}、$e_{B_{11}}$、$e_{B_{12}}\cdots e_{B_{32}}$ 之间的互协因数及协因数;然后利用最小二乘结果组成 $\boldsymbol{K}$ 矩阵,代入公式(7)得到了总体最小二乘平差的随机模型。通过计算可以看出,由于总体最小二乘方法考虑了已知点坐标的误差,得到的平差结果精度比最小二乘更高,单位权中误差精度提高了 94.18%,待定点 D 坐标的中误差精度分别提高了 93.88% 和 94.13%;且待定点 D 坐标的平差值与最小二乘结果也不相同,分别变化了 0.034 m 和 0.015 m。因此,总体最小二乘方法同时顾及系数矩阵及观测值的误差,可以得到更加合理的参数估值,这就是总体最小二乘方法的优势所在。

表 2 平差结果数据表

Tab. 2 Surveying adjustment results

	$\hat{x}_D$ (m)	$\hat{y}_D$ (m)	$\hat{X}_D$ (m)	$\hat{Y}_D$ (m)	$\hat{\sigma}_{x_D}$ (cm)	$\hat{\sigma}_{y_D}$ (cm)	$\hat{\sigma}_0$ (cm)
TLS	0.070	-0.357	2 326.329	5 330.048	0.651	0.695	0.347
LS	0.036	-0.151	2 326.295	5 330.033	10.634	11.224	5.959

5 结束语

总体最小二乘方法可以同时顾及系数矩阵及观测值的误差,因此,可以得到更加合理的参数估值。本文详细推导了观测值与系数矩阵元素相关且精度不等下的总体最小二乘平差公式,并利用矩阵拉直

运算进行随机模型的求解,给出了总体最小二乘平差的精度评定公式。测边网坐标平差中由于已知点含有误差,因而其误差方程的系数矩阵含有误差且与边长观测值相关并精度不等,为了提高待定点坐标平差精度,本文推导了测边网坐标平差的总体最小二乘方法。通过计算分析,表明总体最小二乘结果具有更高的精度和合理性。随着总体最小二乘方法的深入研究及应用,本文推导的观测值与系数矩阵元素相关且精度不等下的总体最小二乘平差方法将发挥作用。

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附录:分别求导结果

系数矩阵中元素 B_{11} 是 L_1、L_2、X_A、Y_A、X_B、Y_B 的函数,因而分别求导有

$$\frac{\partial B_{11}}{\partial L_1} = -\frac{1}{2}\,\frac{\eta_1(X_B - X_A) + (4L_1^2\eta_2 - \eta_1^2)^{1/2}(Y_B - Y_A)}{L_1^2\eta_2} + \frac{1}{2}\,\frac{2L_1(X_B - X_A) + \dfrac{1}{2}\,\dfrac{(Y_B - Y_A)(8L_1\eta_2 - 4\eta_1 L_1)}{(4L_1^2\eta_2 - \eta_1^2)^{1/2}}}{L_1\eta_2}$$

$$\frac{\partial B_{11}}{\partial L_2} = \frac{1}{2}\,\frac{\dfrac{-2L_2(X_B - X_A) + 2(Y_B - Y_A)(L_1^2 + (X_B - X_A)^2 + (Y_B - Y_A)^2 - L_2^2)L_2}{(4L_1^2((X_B - X_A)^2 + (Y_B - Y_A)^2) - (L_1^2 + (X_B - X_A)^2 + (Y_B - Y_A)^2 - L_2^2)^2)^{1/2}}}{L_1((X_B - X_A)^2 + (Y_B - Y_A)^2)}$$

$$\frac{\partial B_{11}}{\partial X_A} = -\frac{1}{2}\frac{\eta_1(X_B-X_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(Y_B-Y_A))(-2X_B+2X_A)}{L_1\eta_2^2}+\frac{1}{2}\{(-2X_B+2X_A)(X_B-X_A)-\eta_1+\frac{1}{2}\frac{(Y_B-Y_A)(4L_1^2(-2X_B+2X_A)-2\eta_1(-2X_B+2X_A))}{(4L_1\eta_2^2-\eta_1^2)^{1/2}}\}/L_1\eta_2$$

$$\frac{\partial B_{11}}{\partial Y_A} = -\frac{1}{2}\frac{\eta_1(X_B-X_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(Y_B-Y_A))(-2Y_B+2Y_A)}{L_1\eta_2^2}+\{\frac{1}{2}(-2Y_B+2Y_A)(X_B-X_A)+\frac{1}{2}[\frac{(Y_B-Y_A)(4L_1^2(-2Y_B+2Y_A)-2\eta_1(-2Y_B+2Y_A))}{(4L_1\eta_2^2-\eta_1^2)^{1/2}}]-4(L_1^2\eta_2-\eta_1^2)^{1/2}\}/L_1\eta_2$$

$$\frac{\partial B_{11}}{\partial X_B} = -\frac{1}{2}\frac{\eta_1(X_B-X_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(Y_B-Y_A))(-2X_B-2X_A)}{L_1\eta_2^2}+\frac{1}{2}\{(2X_B-2X_A)(X_B-X_A)+\eta_1+\frac{1}{2}\frac{(Y_B-Y_A)(4L_1^2(2X_B-2X_A)-2\eta_1(2X_B-2X_A))}{(4L_1\eta_2^2-\eta_1^2)^{1/2}}\}/L_1\eta_2$$

$$\frac{\partial B_{11}}{\partial Y_B} = -\frac{1}{2}\frac{(\eta_1(X_B-X_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(Y_B-Y_A))(2Y_B-2Y_A)}{L_1\eta_2^2}+\{\frac{1}{2}(2Y_B-2Y_A)(X_B-X_A)+\frac{1}{2}[\frac{(Y_B-Y_A)(4L_1^2(2Y_B-2Y_A)-2\eta_1(2Y_B-2Y_A))}{(4L_1\eta_2^2-\eta_1^2)^{1/2}}]+4(L_1^2\eta_2-\eta_1^2)^{1/2}\}/L_1\eta_2$$

式中:$\eta_1=L_1^2+(X_B-X_A)^2+(Y_B-Y_A)^2-L_2^2$,$\eta_2=(X_B-X_A)^2+(Y_B-Y_A)^2$

$$B_{12}=\frac{\Delta Y_{AD}}{S_{AD}}=\frac{1}{2L_1}\frac{1}{\overline{AB}^2}((L_1^2+\overline{AB}^2-L_2^2)(Y_A-Y_B)+(4L_1^2\overline{AB}^2-(L_1^2+\overline{AB}^2-L_2^2)^2)^{\frac{1}{2}}(X_B-X_A)) \tag{24}$$

系数矩阵中元素 B_{12} 是 L_1、L_2、X_A、Y_A、X_B、Y_B 的函数,因而求导有

$$\frac{\partial B_{12}}{\partial L_1} = -\frac{1}{2}\frac{\eta_1(Y_B-Y_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(X_B-X_A)}{L_1^2\eta_2}+\frac{1}{2}\frac{2L_1(Y_B-Y_A)+\frac{1}{2}\frac{(X_B-X_A)(8L_1\eta_2-4\eta_1L_1)}{(4L_1^2\eta_2-\eta_1^2)^{1/2}}}{L_1\eta_2}$$

$$\frac{\partial B_{12}}{\partial L_2} = \frac{1}{2}\frac{\frac{-2L_2(Y_B-Y_A)+2(X_B-X_A)(L_1^2+(X_B-X_A)^2+(Y_B-Y_A)^2-L_2^2)L_2}{(4L_1^2((X_B-X_A)^2+(Y_B-Y_A)^2)-(L_1^2+(X_B-X_A)^2+(Y_B-Y_A)^2-L_2^2)^2)^{1/2}}}{L_1((X_B-X_A)^2+(Y_B-Y_A)^2)}$$

$$\frac{\partial B_{12}}{\partial X_A} = -\frac{1}{2}\frac{(\eta_1(Y_B-Y_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(X_B-X_A))(-2X_B+2X_A)}{L_1\eta_2^2}+\frac{1}{2}\{(-2X_B+2X_A)(Y_B-Y_A)+\frac{1}{2}\frac{(X_B-X_A)(4L_1^2(-2X_B+2X_A)-2\eta_1(-2X_B+2X_A))}{(4L_1\eta_2^2-\eta_1^2)^{1/2}}-(4L_1\eta_2^2-\eta_1^2)^{\frac{1}{2}}\}/L_1\eta_2$$

$$\frac{\partial B_{12}}{\partial Y_A} = -\frac{1}{2}\frac{(\eta_1(Y_B-Y_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(X_B-Y_A))(-2Y_B+2Y_A)}{L_1\eta_2^2}+\{\frac{1}{2}(-2Y_B+2Y_A)(Y_B-Y_A)-\eta_1+\frac{1}{2}[\frac{(X_B-X_A)(4L_1^2(-2Y_B+2Y_A)-2\eta_1(-2Y_B+2Y_A))}{(4L_1\eta_2^2-\eta_1^2)^{1/2}}]\}/L_1\eta_2$$

$$\frac{\partial B_{12}}{\partial X_B} = -\frac{1}{2}\frac{(\eta_1(Y_B-Y_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(X_B-X_A))(-2X_B-2X_A)}{L_1\eta_2^2}+\frac{1}{2}\{(2X_B-2X_A)(Y_B-Y_A)+\frac{1}{2}\frac{(X_B-X_A)(4L_1^2(2X_B-2X_A)-2\eta_1(2X_B-2X_A))}{(4L_1\eta_2^2-\eta_1^2)^{1/2}}+(4L_1\eta_2^2-\eta_1^2)^{\frac{1}{2}}\}/L_1\eta_2$$

$$\frac{\partial B_{12}}{\partial Y_B} = -\frac{1}{2}\frac{(\eta_1(Y_B-Y_A)+(4L_1^2\eta_2-\eta_1^2)^{1/2}(X_B-X_A))(2Y_B-2Y_A)}{L_1\eta_2^2}+\{\frac{1}{2}(2Y_B-2Y_A)(Y_B-Y_A)+\eta_1+\frac{1}{2}[\frac{(X_B-X_A)(4L_1^2(2Y_B-2Y_A)-2\eta_1(2Y_B-2Y_A))}{(4L_1\eta_2^2-\eta_1^2)^{1/2}}]\}/L_1\eta_2$$

式中:$\eta_1=L_1^2+(X_B-X_A)^2+(Y_B-Y_A)^2-L_2^2$,$\eta_2=(X_B-X_A)^2+(Y_B-Y_A)^2$